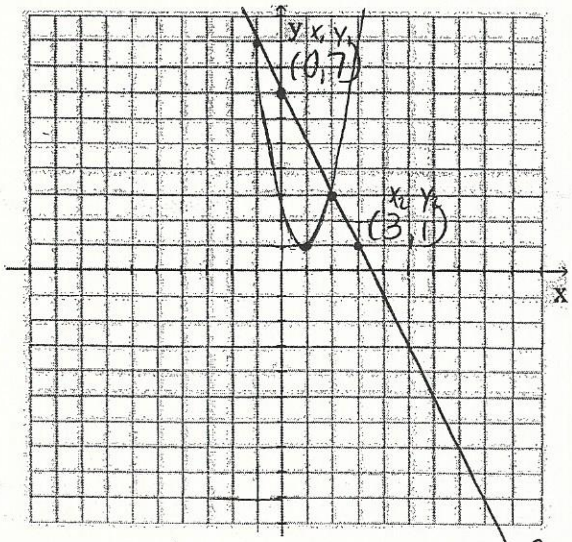
Answer key

March 27, 2013

Word Problems: Solving a System of Equations (Quadratic/ Linear)

1. Consider the line ℓ passing through points A (0,7) and B (3,1) and the parabola with vertex V (1,1) passing through the point C (-1,9). Find the points of intersection P and Q of the parabola and the line.



① Find eqn of line.

y-int: 7

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-7}{3-0} = \frac{-6}{3} = -2$

$\boxed{\text{Eqn: } y_L = -2x + 7}$

② Find eqn of parabola

We have vertex and 1 point → plug into vertex form to get 'a'. $V(\underset{h}{1}, \underset{k}{1})$ $Pt(\underset{x}{-1}, \underset{y}{9})$

$y = a(x-h)^2 + k$

$9 = a(-1-1)^2 + 1$

$9 = a(-2)^2 + 1$

$9 = 4a + 1$

$9 - 1 = 4a$

$\frac{8}{4} = \frac{4a}{4}$

$2 = a$

$\boxed{\text{Eqn: } y = 2(x-1)^2 + 1}$

③ Simplify parabola eqn into $y = ax^2 + bx + c$ form.

$y = 2(x-1)^2 + 1$

$y = 2(x-1)(x-1) + 1$

$y = 2(x^2 - 1x - 1x + 1) + 1$

$y = 2(x^2 - 2x + 1) + 1$

$y = 2x^2 - 4x + 2 + 1$

$\boxed{y_P = 2x^2 - 4x + 3}$

Ans: ________

④ Same point so

$y_P = y_L$

$2x^2 - 4x + 3 = -2x + 7$

$2x^2 - 4x + 2x + 3 - 7 = 0$

$\underset{a}{2x^2} \underset{b}{- 2x} \underset{c}{- 4} = 0$

⑤ Plug a, b, c into quadratic equation

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-4)}}{2(2)}$

$= \frac{2 \pm \sqrt{4 + 32}}{4} = \frac{2 \pm \sqrt{36}}{4}$

$= \frac{2 \pm 6}{4} \rightarrow \frac{2+6}{4} = \frac{8}{4} = \overset{x}{\boxed{2}}$

$\rightarrow \frac{2-6}{4} = \frac{-4}{4} = \overset{x}{\boxed{-1}}$

⑥ Plug both x-coordinates into either eqn to find 'y'.

$y_L = -2x + 7 = -2(\overset{x}{2}) + 7 = -4 + 7 = \overset{y}{\boxed{3}}$

$y_L = -2x + 7 = -2(\overset{x}{-1}) + 7 = 2 + 7 = \overset{y}{\boxed{9}}$

$\boxed{\text{Ans: } (2,3) \ \& \ (-1,9)}$

1) Find Mohammed's eqn.
y-int: 54

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{72-54}{9-0} = \frac{18}{9} = \boxed{2}$

$\boxed{\text{Eqn: } y_L = 2x + 54}$

2) Find Malik's equation $V(\underset{h}{10}, \underset{k}{80})$ $P(\underset{x}{0}, \underset{y}{30})$
→ Have vertex & 1 point → vertex form.

March 27, 2013

$y = a(x-h)^2 + k$
$30 = a(0-10)^2 + 80$
$30 = a(-10)^2 + 80$
$30 = 100a + 80$
$30 - 80 = 100a$

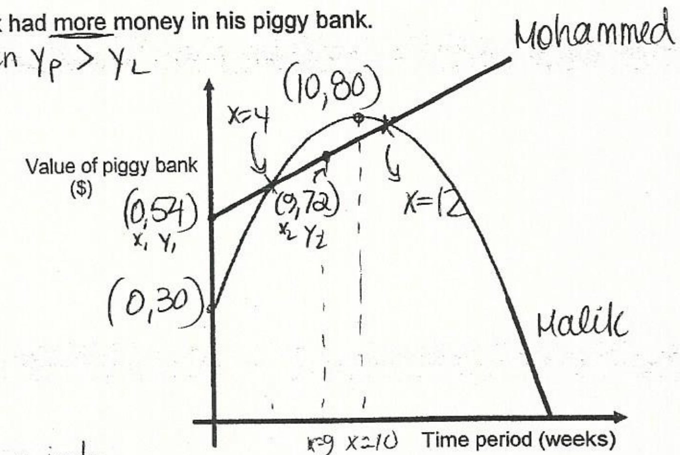
$\frac{-50}{100} = \frac{100a}{100}$

$\boxed{\frac{-1}{2} = a}$

$\boxed{\text{Eqn: } y = \frac{-1}{2}(x-10)^2 + 80}$

2. Mohammed and Malik each decided to start saving their spare change in their piggy banks. They agreed to do so for 24 weeks to see who is the better saver.
The change in the value of Malik's piggy bank is represented by a parabola. Malik started his savings by putting $30 in his bank. After 10 weeks Malik's piggy reached its maximum value of $80.
Mohammed appears to have been a more consistent saver. He started his savings with an initial investment of $54, and after 9 weeks his piggy bank value had increased by $\frac{1}{3}$. → $54\left(\frac{1}{3}\right) = 18$ → new total $= 54 + 18 = 72$

Determine the number of weeks that Malik had more money in his piggy bank.
$= x$ when $y_P > y_L$



3) Simplify y_P into $y = ax^2 + bx + c$ form.

$y = \frac{-1}{2}(x-10)^2 + 80$
$y = \frac{-1}{2}(x-10)(x-10) + 80$
$y = \frac{-1}{2}(x^2 - 10x - 10x + 100) + 80$
$y = \frac{-1}{2}(x^2 - 20x + 100) + 80$
$y = \frac{-1}{2}x^2 + 10x - 50 + 80$
$\boxed{y_P = \frac{-1}{2}x^2 + 10x + 30}$

4) Let $y_P = y_L$
$\frac{-1}{2}x^2 + 10x + 30 = 2x + 54$
$\frac{-1}{2}x^2 + 10x - 2x + 30 - 54 = 0$
$\underset{a}{\frac{-1}{2}x^2} + \underset{b}{8x} - \underset{c}{24} = 0$

5) Plug a, b, c into quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{(8)^2 - 4(-0.5)(-24)}}{2\left(\frac{-1}{2}\right)} = \frac{-8 \pm \sqrt{64 - 48}}{-1}$

$= \frac{-8 \pm \sqrt{16}}{-1} = \frac{-8 \pm 4}{-1}$ → $\frac{-8+4}{-1} = \frac{-4}{-1} = \boxed{4}$ (x)

↘ $\frac{-8-4}{-1} = \frac{-12}{-1} = \boxed{12}$ (x)

6) Look at graph again
Difference in x-coordinates where $y_P > y_L$ is $12 - 4$
$= 8$ weeks

Ans: 8 weeks

3. Two competing fitness centres are each having a promotion to attract new members. The rule $y = \frac{1}{4}x^2 + 50$ gives the number of members at club A and the rule $y = 10x - 25$ gives the number of members at club B since the start of the their promotions ($x \geq 3$).

$Y_L = 10x - 25$

x	y
3	5
4	15
5	25

a) Represent this situation in the Cartesian plane given below.

$Y_P = \frac{1}{4}x^2 + 50$

x	y
0	50
2	51
4	54
6	59
8	66
10	75
12	86
14	99

$y = \frac{1}{4}(4)^2 + 50$

$y = \frac{1}{4}(16) + 50$

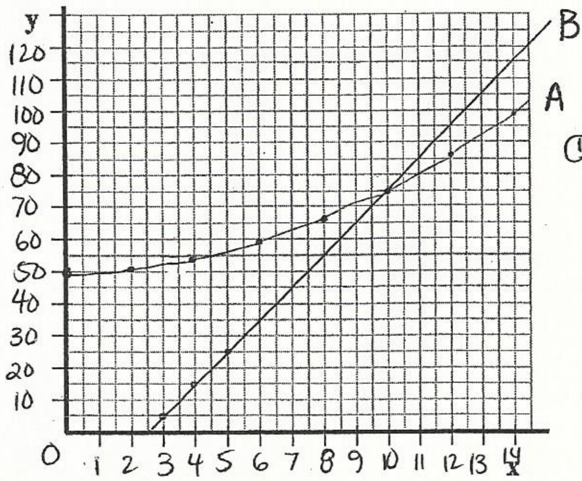
$y = 4 + 50$

$y = 54$

$y = \frac{1}{4}(8)^2 + 50$

$y = \frac{1}{4}(64) + 50$

$y = 66$



① $Y_P = \frac{1}{4}x^2 + 50 \quad Y_L = 10x - 25$

② $Y_P = Y_L$

③ $\frac{1}{4}x^2 + 50 = 10x - 25$

$\frac{1}{4}x^2 - 10x + 50 + 25 = 0$

$0.25x^2 - 10x + 75 = 0$

a b c

b) How many days after the start of the promotion do the two centres have the same number of members?

④ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(0.25)(75)}}{2(0.25)}$

$= \frac{10 \pm \sqrt{100 - 75}}{0.5} = \frac{10 \pm \sqrt{25}}{0.5} = \frac{10 \pm 5}{0.5}$

$\rightarrow \frac{10+5}{0.5} = \frac{15}{0.5} = \boxed{30}$

$\frac{10-5}{0.5} = \frac{5}{0.5} = \boxed{10}$

⑤ Remember, the promotion starts at Day 3 → so the two centres actually had the same number of members (30−3=) 27 days and (10−3=) 7 days after the start of the promotion

Ans: 7 and 27 days

c) How many days after the start of the promotion does centre B have as many or more members than centre A?

Centre B has more members than Centre A for 20 days in total, from day 7 to 27.

Ans: ________

March 27, 2013

① Math club eqn:

$y = 10$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15-10}{10-0} = \frac{5}{10} = \boxed{\frac{1}{2}}$

$\boxed{\text{Eqn: } y_L = \frac{1}{2}x + 10}$

② French club eqn:

Vertex form: $(10, 3)$, Pt$(0, 28)$ — h, k; x, y

$y = a(x-h)^2 + k$

$28 = a(0-10)^2 + 3$

$28 = a(-10)^2 + 3$

$28 = 100a + 3$

$28 - 3 = 100a$

$\frac{25}{100} = \frac{100a}{100}$

$\boxed{1/4 = a}$

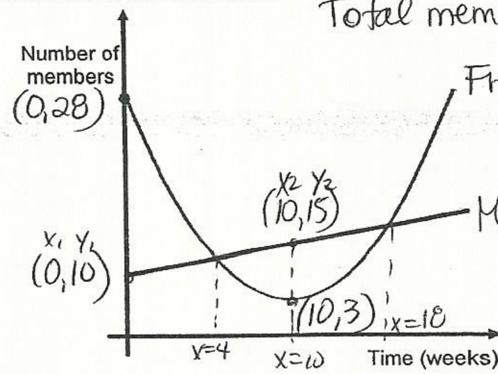
$\boxed{\text{Eqn: } y = \frac{1}{4}(x-10)^2 + 3}$

4. The Math and French clubs tracked their numbers of members over a 20-week period. The number of members of the French club is represented by a parabola. At the beginning the French club had 28 members, and its membership reached its minimum at 10 weeks, with only 3 members.

The Math club membership is represented by a straight line. The initial number of members was 10. After 10 weeks the membership had increased by 50%. → 10 members x 50% = 5 more members

Total members = 10 + 5 = 15 memb

Determine the number of weeks in which the Math club had more members than the French club. x only.



③ Simplify y_P into $y = ax^2 + bx + c$

$y = \frac{1}{4}(x-10)^2 + 3$

$y = \frac{1}{4}(x-10)(x-10) + 3$

$y = \frac{1}{4}(x^2 - 10x - 10x + 100) + 3$

$y = \frac{1}{4}(x^2 - 20x + 100) + 3$

$y = \frac{1}{4}x^2 - 5x + 25 + 3$

$\boxed{y_P = \frac{1}{4}x^2 - 5x + 28}$

④ $y_P = y_L$

⑤ $\frac{1}{4}x^2 - 5x + 28 = \frac{1}{2}x + 10$

$0.25x^2 - 5x - 0.5x + 28 - 10 = 0$

$\underset{a}{0.25}x^2 \underset{b}{- 5.5}x \underset{c}{+ 18} = 0$

⑥ Quadratic eqn:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5.5) \pm \sqrt{(-5.5)^2 - 4(0.25)(18)}}{2(0.25)}$

$= \frac{5.5 \pm \sqrt{30.25 - 18}}{0.5} = \frac{5.5 \pm \sqrt{12.25}}{0.5}$

$= \frac{5.5 \pm 3.5}{0.5} \to \frac{5.5 + 3.5}{0.5} = \frac{9}{0.5} = \boxed{18}$ (x)

$\frac{5.5 - 3.5}{0.5} = \frac{2}{0.5} = \boxed{4}$ (x)

⑦ Math club had more members between 4 and 18 weeks

$\to 18 - 4 = \boxed{14 \text{ weeks}}$

Ans: 14 weeks

March 27, 2013

① Spotlight eq'n:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{20-0}{93-68} = \frac{20}{25} = \boxed{\frac{4}{5}}$

$y = mx + b$?

$0 = \frac{4}{5}(68) + b$

$0 = 54.4 + b$

Plug into $y=mx+b$ to find b: $\boxed{-54.4 = b}$

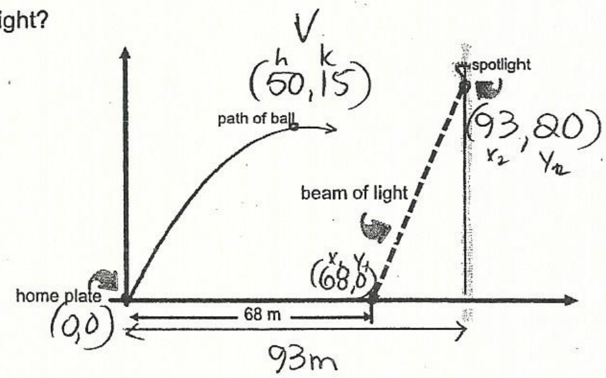
$\boxed{\text{Eqn: } y_L = 0.8x - 54.4}$

5. A batter hits a baseball from home plate. The path of the ball describes a parabolic trajectory. When the ball reaches its maximum height, 15 metres, it has travelled a horizontal distance of 50 metres from home plate.

A spotlight which is 20 metres above the ground, and 93 metres from home plate shines a beam of light onto the field. This beam hits the ground a distance of 68 metres from home plate.

At what height will the ball cross the beam of light?

= y at p.o.i.



② Ball eqn:

Vertex form → $V(50,15)$, $Pt(0,0)$ (h k; x y)

$y = a(x-h)^2 + k$

$0 = a(0-50)^2 + 15$

$0 = a(-50)^2 + 15$

$0 = 2500a + 15$

$\frac{-15}{2500} = \frac{2500a}{2500}$

$\boxed{-0.006 = a}$

$\boxed{\text{Eqn: } y_P = -0.006(x-50)^2 + 15}$

③ Simplify y_P into $y = ax^2 + bx + c$

$y_P = -0.006(x-50)^2 + 15$

$y_P = -0.006(x-50)(x-50) + 15$

$y_P = -0.006(x^2 - 50x - 50x + 2500) + 15$

$y_P = -0.006(x^2 - 100x + 2500) + 15$

$y_P = -0.006x^2 + 0.6x \cancel{-15} \cancel{+15}$

$\boxed{y_P = -0.006x^2 + 0.6x}$

Ans: ________

④ $y_P = y_L$ at intersection

$-0.006x^2 + 0.6x = 0.8x - 54.4$

$-0.006x^2 + 0.6x - 0.8x + 54.4 = 0$

$-0.006x^2 - 0.2x + 54.4 = 0$

a b c

⑤ Quadratic equation:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-0.2) \pm \sqrt{(-0.2)^2 - 4(-0.006)(54.4)}}{2(-0.006)}$

$= \frac{0.2 \pm \sqrt{0.04 + 1.3056}}{-0.012} = \frac{0.2 \pm \sqrt{1.3456}}{-0.012}$

$= \frac{0.2 \pm 1.16}{-0.012} \rightarrow \frac{0.2 + 1.16}{-0.012} = \frac{1.36}{-0.012} = \cancel{-113.3}$ can't have a negative dist

$\frac{0.2 - 1.16}{-0.012} = \frac{-0.96}{-0.012} = \boxed{80}$

x

⑥ Pick an eq'n and plug 'x' in to get 'y':

$y_L = 0.8x - 54.4 = 0.8(80) - 54.4 = 64 - 54.4$

$= 9.6m$

Ans: The ball will cross the beam of light at 9.6m.

6. Two projectiles are launched simultaneously. The heights (in m), h_1 (t) and h_2 (t) of the two projectiles as a function of time (t) in seconds since their launch are given by the rules h_1 (t) = $-2t^2 + 20t + 50$ and h_2 (t) = $2t + 66$.

a) How long after their launch will the two projectiles be at the same height? = intersection
= x

b) Over what interval of time since the launch is the 1st projectile higher than the 2nd?
= x

① $h_1 = h_2$

$-2t^2+20t+50=2t+66$

$-2t^2+20t-2t+50-66=0$

$-2t^2+18t-16=0$

a b c

② $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-18\pm\sqrt{(18)^2-4(-2)(-16)}}{2(-2)}$

$=\frac{-18\pm\sqrt{324-128}}{-4}=\frac{-18\pm\sqrt{196}}{-4}$

$=\frac{-18\pm14}{-4}\rightarrow\frac{-18+14}{-4}=\frac{-4}{-4}=\boxed{1}$ x

$\frac{-18-14}{-4}=\frac{-32}{-4}=\boxed{8}$ x

a) Ans: They will be at the same height 1 sec after the launch, and again at 8 sec after the launch

b) One projectile is higher than the second from 1 sec to 8sec

→ 8-1 = 7sec.

b) Ans: One projectile will be higher than the second for 7 seconds.

a) Ans: ________

b) Ans: ________

March 27, 2013

7. A study involved looking at the population changes for two competing animal species in a closed environment. The population change for species A is described by a quadratic function. At the beginning of the study, this species consisted of 145 individuals. By the eleventh month, the population of this species had reached a peak of 210 individuals.

At the beginning of the study, species B consisted of 125 individuals. Six months later, this number had risen by 20%. The population change for species B is described by a linear function. During which month did the population of species B exceed that of species A? $= x \ (>0)$

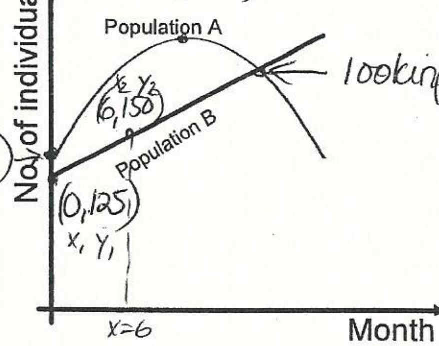
$125 \times 20\% = 25$ more

$125 + 25 = 150$ total

Clearly show all your work.

V

(11, 210)



looking for x.

(0,145)

① Pop. B: $y = mx + b$

y-int = 125

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{150-125}{6-0} = \boxed{\frac{25}{6}}$ or 4.167

$\boxed{\text{Eq'n: } y_L = 4.167x + 125}$

② Pop. A Vertex eq'n.

$V \to (11, 210)$, Pt $(0, 145)$

h k x y

$y = a(x-h)^2 + k$

$145 = a(0-11)^2 + 210$

$145 = a(-11)^2 + 210$

$145 = 121a + 210$

$145 - 210 = 121a$

$\frac{-65}{121} = \frac{121a}{121}$

$\boxed{-0.537 = a}$

$\boxed{\text{Eq'n: } y_P = -0.537(x-11)^2 + 210}$

③ Simplify y_P into $y = ax^2 + bx + c$

$y = -0.537(x-11)^2 + 210$

$y = -0.537(x-11)(x-11) + 210$

$y = -0.537(x^2 - 11x - 11x + 121) + 210$

$y = -0.537(x^2 - 22x + 121) + 210$

$y = -0.537x^2 + 11.818x - 64.977 + 210$

$y = -0.537x^2 + 11.818x + 145.023$

a b c

④ $y_P = y_L$

$-0.537x^2 + 11.818x + 145.023 = 4.167x + 125$

$-0.537x^2 + 11.818x - 4.167x + 145.023 - 125 = 0$

$-0.537x^2 + 7.651x + 20.023 = 0$

a b c

⑤ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7.651 \pm \sqrt{(7.651)^2 - 4(-0.537)(20.023)}}{2(-0.537)}$

$= \frac{-7.651 \pm \sqrt{58.538 + 43.009}}{-1.074} = \frac{-7.651 \pm \sqrt{101.547}}{-1.074}$

$= \frac{-7.651 \pm 10.077}{-1.074} \to \frac{-7.651 + 10.077}{-1.074} = \frac{2.426}{-1.074} = \cancel{-2.26}$

$\to \frac{-7.651 - 10.077}{-1.074} = \frac{-17.728}{-1.074} = 16.51$

= after the 16th month, so during the 17th month

Ans: During the 17th month

1) Light ray.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12 - 8.25}{21 - 18} = \frac{3.75}{3} = \boxed{1.25}$

$12 = (1.25)(21) + b$

$12 = 26.25 + b$

$12 - 26.25 = \boxed{b = -14.25}$

$\boxed{\text{Eqn: } y_L = 1.25x - 14.25}$

March 27, 2013

8. A parabolic mirror is represented in the system below, where the units are measured in $\boxed{\text{centimetres.}}$ The parabola passes through point (8, 7.75) and its vertex is point (21, 3). (V)

x y

h k

A light ray, which is parallel to the y-axis, strikes the mirror and is reflected in a straight line passing through points (18, 8.25) and (21, 12).

x_1 y_1 x_2 y_2

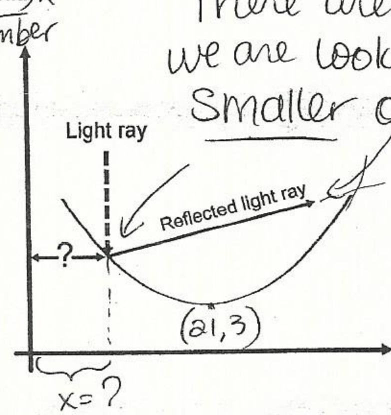
= x

At what distance from the y-axis does the light ray strike the parabolic mirror? Round off your answer to the nearest unit. *

whole number

Clearly show all your work.

There are 2 p.o.i.s – we are looking for the Smaller one (closer to y-axis).



2) Mirror

Vertex form: V(21,3), Pt(8, 7.75)

h k x y

$y = a(x-h)^2 + k$

$7.75 = a(8-21)^2 + 3$

$7.75 = a(-13)^2 + 3$

$7.75 = 169a + 3$

$7.75 - 3 = 169a$

$\frac{4.75}{169} = \frac{169a}{169}$

$\boxed{0.0281 = a}$

$\boxed{\text{Eqn: } y_P = 0.0281(x-21)^2 + 3}$

③ Simplify y_P into $y = ax^2 + bx + c$ form.

$y = 0.0281(x-21)^2 + 3$

$y = 0.0281(x-21)(x-21) + 3$

$y = 0.0281(x^2 - 21x - 21x + 441) + 3$

$y = 0.0281(x^2 - 42x + 441) + 3$

$y = 0.0281x^2 - 1.1802x + 12.3921 + 3$

$\boxed{y_P = 0.0281x^2 - 1.1802x + 15.3921}$

④ $y_P = y_L$

Ans: ____________

$0.0281x^2 - 1.1802x + 15.3921 = 1.25x - 14.25$

$0.0281x^2 - 1.1802x - 1.25x + 15.3921 + 14.25 = 0$

$0.0281x^2 - 2.4302x + 29.6421 = 0$

a b c

⑤ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2.4302) \pm \sqrt{(-2.4302)^2 - 4(0.0281)(29.6421)}}{2(0.0281)}$

$= \frac{2.4302 \pm \sqrt{5.9058 - 3.3318}}{0.0562} = \frac{2.4302 \pm \sqrt{2.574}}{0.0562}$

$= \frac{2.4302 \pm 1.6044}{0.0562} \rightarrow \frac{2.4302 + 1.6044}{0.0562} = 71.79$

$\frac{2.4302 - 1.6044}{0.0562} = 14.69$

$\rightarrow 15\text{cm}$

Ans: The light ray will strike the parabolic mirror at a distance of 15 cm.